THE C*-ENVELOPE OF A SEMICROSSED PRODUCT AND NEST REPRESENTATIONS

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ABSTRACT. Let X be compact Hausdorff, and $\varphi : X \to X$ a continuous surjection. Let $\mathcal{A}$ be the semicrossed product algebra corresponding to the relation $fU = Uf \circ \varphi$. Then the C*-envelope of $\mathcal{A}$ is the crossed product of a commutative C*-algebra which contains $C(X)$ as a subalgebra, with respect to a homeomorphism which we construct. We also show there are "sufficiently many" nest representations.

1. INTRODUCTION

In [11] the notion of the semi-crossed product of a C*-algebra with respect to an endomorphism was introduced. This agreed with the notion of a nonselfadjoint or analytic crossed product introduced earlier by McAsey and Muhly ([8]) in the case the endomorphism was an automorphism. Neither of those early papers dealt with the fundamental question of describing the C*-envelopes of the class of operator algebras being considered.

That open question was breached in the paper [9], in which Muhly and Solel described the C*-envelope of a semicrossed product in terms of C*-correspondences, and indeed determined the C*-envelopes of many classes of nonselfadjoint operator algebras.

While it is not our intention to revisit the results of [9] in any detail, we recall briefly what was done. Given a C*-algebra $\mathcal{C}$ and an endomorphism α of $\mathcal{C}$ one forms the semicrossed product $\mathcal{A} := \mathcal{C} \rtimes_\alpha \mathbb{Z}^+$ as described in Section 3. First one views $\mathcal{C}$ as a C*-correspondence $\mathcal{E}$ by taking $\mathcal{E} = \mathcal{C}$ as a right $\mathcal{C}$ module, and the left action given by the endomorphism. One then identifies the tensor algebra (also called the analytic Toeplitz algebra) $\mathfrak{T}_+(\mathcal{E})$ with the semicrossed product $\mathcal{A}$. The C*-envelope of $\mathcal{A}$ is given by the Cuntz-Pimsner algebra $\mathfrak{O}(\mathcal{E})$.

The question that motivated this paper was to find the relation between the C*-envelopes of semicrossed products, and crossed products. Specifically, when is the C*-envelope of a semicrossed product a crossed product? If the endomorphism α of $\mathcal{C}$ is actually an automorphism,

then the crossed product $\mathcal{C} \rtimes_\alpha \mathbb{Z}$ is a natural candidate for the C*-envelope, and indeed, as noted in [9], this is the case. In this paper we answer that question in case the C*-algebra $\mathcal{C}$ is commutative (and unital). Indeed, it turns out that the C*-envelope is *always* a crossed product (cf Theorem 4).

For certain classes of nonselfadjoint operator algebras, nest representations play a fundamental role akin to that of the irreducible representations in the theory of C*-algebras. The notion of nest representation was introduced by Lamoureux ([6], [7]) in a context with similarities to that here. We do not answer the basic question as to whether nest representations suffice for the kernel-hull topology; i.e., every closed ideal in a semicrossed product is the intersection of the kernels of the nest representations containing it. What we do show is that nest representations suffice for the norm: the norm of an element is the supremum of the norms of the isometric covariant nest representations (Theorem 2). The results on nest representation require some results in topological dynamics, which, though not deep, appear to be new.

The history of work in anaylytic crossed products and semicrosed products goes back nearly forty years. While in this note we do not review the literature of the subject, we mention the important paper [3] in which the Jacobson radical of a semicrossed product is determined and necessary and sufficient conditions for semi-simplicity of the crossed product are obtained. We use this in Proposition 3 to show that the simplicity of the C*-envelope implies the semisimplicity of the semicrossed product.

In very recent work of Davidson and Katsoulis ([2]), semicrossed products are viewed as an example of a more general class of Banach Algebras associated with dynamical systems which they call conjugacy algebras. They have extracted fundamental properties needed to obtain, for instance, the result that conjugacy of dynamical systems is equivalent to isomorphism of the conjugacy algebras. It would be worthwhile to extend the results here to the broader context.

2. Dynamical Systems

In our context, X will denote a compact Hausdorff space. By a *dynamical system* we will simply mean a space X together with a mapping $\varphi : X \to X$. In this article, the map φ will always be a continuous surjection.

Definition 1. Given a dynamical system (X, φ) we will say (following the terminology of [12]) the dynamical system (Y, ψ) is an *extension* of (X, φ) in case there is a continuous surjection $p : Y \to X$ such that the

the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \psi\ } & Y \\
{\scriptstyle p}\big\downarrow & & {\scriptstyle p}\big\downarrow \qquad (\dagger) \\
X & \xrightarrow{\ \varphi\ } & X
\end{array}
$$

commutes. The map p is called the extension map (of Y over X).

Notation. In case p is a homeomorphism, it is called a *conjugacy.*

Given a dynamical system (X, φ) there is a canonical procedure for producing an extension (Y, ψ) in which ψ is a homeomorphism.

Let $\tilde{X} = \{(x_1, x_2, \dots) : \ x_n \in X \text{ and } x_n = \varphi(x_{n+1}), \ n = 1, 2, \dots \}$. As $\tilde{X}$ is a closed subset of the product $\Pi_{n=1}^{\infty} X_n$ where $X_n = X, \ n = 1, 2, \dots$, so $\tilde{X}$ is compact Hausdorff. Define a map $\tilde{\varphi} : \tilde{X} \to \tilde{X}$ by

$$\tilde{\varphi}(x_1, x_2, \dots) = (\varphi(x_1), x_1, x_2, \dots).$$

This is continuous, and has an inverse given by

$$\tilde{\varphi}^{-1}(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

Define a continuous surjection $p : \tilde{X} \to X$ by

$$p(x_1, x_2, \dots) = x_1.$$

With the map p, the system $(\tilde{X}, \tilde{\varphi})$ is an extension of the dynamical system (X, φ) in which the dynamics of the extension is given by a homeomorphism.

Definition 2. In the case of an extension in which the dynamics is given by a homeomorphism, we will say the extension is a *homeomorphism extension.*

Notation. We will call the extension $(\tilde{X}, \tilde{\varphi})$ the *canonical* homeomorphism extension. If $\tilde{x} \in \tilde{X}$, $\tilde{x} = (x_1, x_2, \dots)$, we will say that $(x_1, x_2, \dots)$ are the *coordinates* of $\tilde{x}$.

Definition 3. Given a dynamical system (X, φ), a homeomorphism extension (Y, ψ) is said to be *minimal* if, whenever (Z, σ) has the property that it is a homeomorphism extension of (X, φ), and (Y, ψ) is an extension of (Z, σ) such that the composition of the extension maps of Z over X with the extension map of Y over Z is the extension map of Y over X, then (Y, ψ) and (Z, σ) are conjugate.

Lemma 1. *Let (X, φ) be a dynamical system. Then the canonical homeomorphism extension $(\tilde{X}, \tilde{\varphi})$ is minimal.*

Proof. Suppose (Z, σ) is a homeomorphism extension of (X, φ), $p :$ $\tilde{X} \to Z$ and $q : Z \to X$ are continuous surjections, and the diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{X} \\
p \downarrow & & p \downarrow \\
Z & \xrightarrow{\sigma} & Z \\
q \downarrow & & q \downarrow \\
X & \xrightarrow{\varphi} & X
\end{array}
$$

commutes and the composition $q \circ p$ is the extension map of $\tilde{X}$ over X, i.e., the projection onto the first coordinate.

Observe that the canonical homeomorphism extension $(\tilde{Z}, \tilde{\psi})$ of (Z, ψ) is in fact conjugate to (Z, ψ). Indeed, the map $z \in Z \mapsto (z, \psi^{-1}(z), \psi^{-2}(z), \dots)$ is a conjugacy. Thus it is enough to show that $(\tilde{X}, \tilde{\varphi})$ is conjugate to $(\tilde{Z}, \tilde{\sigma})$.

Define a map $r : \tilde{Z} \to \tilde{X}$ by

$$\tilde{z} := (z, \sigma^{-1}(z), \sigma^{-2}(z), \dots) \in \tilde{Z} \mapsto \tilde{x} := (q(z), q(\sigma^{-1}(z), q(\sigma^{-2}(z)), \dots).$$

Observe that this maps into $\tilde{X}$, since

$$\varphi(q(\sigma^{-(n+1)}(z))) = q(\sigma(\sigma^{-(n+1)}(z)))$$
$$= q(\sigma^{-n}(z))$$

Next we claim r maps onto $\tilde{X}$. Let $\tilde{x} = (x_1, x_2, \dots)$ be any element of $\tilde{X}$. Let $z_n \in Z$ be any element such that $q(z_n) = x_n$, $n = 1, 2, \dots$. Let

$$\tilde{z}_n := (\sigma^{n-1}(z_n), \dots, \underset{n}{z_n}, \sigma^{-1}(z_n), \dots).$$

A subsequence of $\{\tilde{z}_n\}$ converges, say, to $\tilde{z}$. Since $r(\tilde{z}_m)$ agrees with $\tilde{x}$ in the first n coordinates for all $m \geq n$, it follows that $r(\tilde{z}) = \tilde{x}$.

To show that r is one-to-one, define a map $\tilde{p} : \tilde{X} \to \tilde{Z}$ by

$$\tilde{p}(\tilde{x}) = (p(\tilde{x}), \sigma^{-1} \circ p(\tilde{x}), \sigma^{-2} \circ p(\tilde{x}) \dots).$$

Note that the fact that $p : \tilde{X} \to Z$ is surjective implies that $\tilde{p}$ is surjective. Let $\tilde{x} = (x_1, x_2, x_3, \dots) \in \tilde{X}$. Then

$$r \circ \tilde{p}(\tilde{x}) = r(p(\tilde{x}),\ \sigma^{-1} \circ p(\tilde{x}),\ \sigma^{-2} \circ p(\tilde{x}), \dots)$$
$$= (q \circ p(\tilde{x}),\ q \circ \sigma^{-1} \circ p(\tilde{x}),\ q \circ \sigma^{-2} \circ p(\tilde{x}), \dots)$$
$$= (x_1,\ q \circ p \circ \tilde{\varphi}^{-1}(\tilde{x}),\ q \circ p \circ \tilde{\varphi}^{-2}(\tilde{x}), \dots)$$
$$= (x_1, x_2, x_3, \dots) = \tilde{x}.$$

where we have used the fact that $q \circ p$ is the projection onto the first coordinate of $\tilde{x}$. Since $\tilde{p}$ is surjective and $r \circ \tilde{p}$ is injective, it follows that r is injective, and hence r is a conjugacy.

$\square$

Lemma 2. *Let (X, φ) be a dynamical system, and let (Y, ψ) be a minimal homeomorphism extension. Then (Y, ψ) is conjugate to the canonical homeomorphism extension, $(\tilde{X}, \tilde{\varphi})$.*

Proof. By assumption there is a continuous surjection $p : Y \to X$ such that the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\psi} & Y \\
\big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle q} \\
X & \xrightarrow{\varphi} & X
\end{array}
$$

commutes.

Consider the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\psi} & Y \\
\big\downarrow{\scriptstyle \tilde{q}} & & \big\downarrow{\scriptstyle \tilde{q}} \\
\tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{X} \\
\big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\
X & \xrightarrow{\varphi} & X
\end{array}
$$

where p denotes the canonical extension map of $\tilde{X}$ over X (i.e., the projection onto the first coordinate), and the map $\tilde{q}$ is defined as follows:

$$\text{For } y \in Y, \ \tilde{q}(y) = (q(y), q \circ \psi^{-1}(y), q \circ \psi^{-2}(y), \dots).$$

Note that the image lies in $\tilde{X}$ since $\varphi(q \circ \psi^{-(n+1)}(y) = q \circ \psi \circ \psi^{-(n+1)}(y) = q \circ \psi^{-n}(y)$.

Next, observe that $p \circ \tilde{q}(y) = q(y)$, so the extension property is satisfied. Hence, by definition of minimality of the homeomorphism extension (Y, ψ), the map $\tilde{q}$ is a conjugacy.

$\square$

Corollary 1. *Let (X, φ) be a dynamical system. Then there exists a minimal homeomorphism extension (Y, ψ) which is unique up to conjugacy. In particular, the canonical extension $(\tilde{X}, \tilde{\varphi})$ is such a homeomorphism extension.*

If (X, φ) is a dynamical system, then the map $\alpha : C(X) \to C(X)$, $f \mapsto f \circ \varphi$, is a $*$-endomorphism. α is a $*$-automorphism iff φ is a homeomorphism. We can dualize the preceding results as follows:

Corollary 2. *Given a dynamical system (X, φ), there is a minimal commutative C^*-algebra $C(\tilde{X})$ with $*$ automorphism $\tilde{\alpha}$ admitting an embedding $\iota : C(X) \hookrightarrow C(\tilde{X})$ such that $\tilde{\alpha} \circ \iota = \iota \circ \alpha$. Furthermore, this commutative C^*-algebra is unique up to isomorphism.*

Proof. Consider the inductive limit

$$C(X) \xrightarrow{\alpha} C(X) \xrightarrow{\alpha} C(X) \xrightarrow{\alpha} \ldots.$$

The inductive limit is a C*-algebra, $C(Y)$ containing $C(X)$ as a subalgebra, and $C(Y)$ admits an automorphism, β satisfying $\beta(f) = \alpha(f)$ for $f \in C(X)$.

But, with $(\tilde{X}, \tilde{\varphi})$ the minimal homeomorphism extension of (X, φ), and viewing $C(X) \hookrightarrow C(\tilde{X})$, we can consider the inductive limit

$$C(X) \xrightarrow{id} \tilde{\alpha}^{-1}(C(X)) \xrightarrow{id} \tilde{\alpha}^{-2}(C(X)) \ldots.$$

The two inductive limits are isometrically isomorphic, as we have the commutative diagram

$$
\begin{array}{ccc}
C(X) & \xrightarrow{\tilde{\alpha}^{-n}} & \tilde{\alpha}^{-n}(C(X)) \\
\alpha \downarrow & & id \downarrow \\
C(X) & \xrightarrow{\tilde{\alpha}^{-(n+1)}} & \tilde{\alpha}^{-(n+1)}(C(X)))
\end{array}
$$

Thus, we may identify Y with $\tilde{X}$, and we have the relation

$$\tilde{\alpha}^{-(n+1)}\alpha(f) = \tilde{\alpha}^{-n}(f), \ f \in C(X), \ n \in \mathbb{Z}^+,$$

hence

$$\alpha(f) = \tilde{\alpha}(f), \ \text{or} \ \tilde{\alpha} \circ \iota = \iota \circ \alpha$$

if we denote the embedding of $C(X)$ into $C(\tilde{X})$ by ι.

$\square$

Definition 4. Given a dynamical system (X, φ), a point $x \in X$ is *periodic* if, for some $n \in \mathbb{N}$, $n \geq 1$, $\varphi^n(x) = x$. If n is the smallest integer with this property, we say that x is periodic of period n. If x is not periodic, we say x is *aperiodic*. If for some $m \in \mathbb{N}$, $\varphi^m(x)$ is periodic, then we say x is *eventually periodic*.

Remark 1. If φ is a homeomorphism, then a point is eventually periodic iff it is periodic; but if φ is a continuous surjection, it is possible to have a point x which is aperiodic and eventually periodic.

Lemma 3. *Let (X, φ) be a dynamical system, and $(\tilde{X}, \tilde{\varphi})$ its minimal homeomorphism extension. Then a point $\tilde{x} = (x_1, x_2, \dots) \in \tilde{X}$ is aperiodic iff for any $n \in \mathbb{N}$, $x_n = x_m$ for at most finitely many $m \in \mathbb{N}$, and $\tilde{x}$ is periodic iff x is periodic.*

Proof. $\tilde{x}$ is periodic of period p iff $\tilde{\varphi}^p(\tilde{x}) = \tilde{x}$, equivalently,

$$(x_1, x_2, \dots) = (\varphi^p(x_1), \varphi^p(x_2), \dots)$$
$$= (x_1, \dots, x_p, x_1, \dots, x_p, \dots)$$

which uses the relation that $\varphi^p(x_{p+j}) = x_j$, $j \in \mathbb{N}$.

This shows that if $\tilde{x}$ is periodic, the coordinates of $\tilde{x}$ form a periodic sequence; the converse is also clear. □

Definition 5.   (1) Recall a dynamical system (X, φ) is *topologically transitive* if for any nonempty open set $\mathcal{O} \subset X$, $\cup_{n=0}^{\infty} \varphi^{-n} \mathcal{O} = X$.
  (2) A dynamical system (X, φ) is *minimal* if there is no proper, closed subset $Z \subset X$ such that $\varphi(Z) = Z$.
  (3) A point x in a dynamical system (X, φ) is *recurrent* if there is a subsequence $\{n_i\}$ of $\mathbb{N}$ such that $\varphi^{n_i}(x) \to x$.

Remark 2. There should be no confusion between the two distinct uses of *minimal*.

Theorem 1. *Let (X, φ) be a dynamical system, and $(\tilde{X}, \tilde{\varphi})$ the minimal homeomorphism extension.*
  (1) *X is metrizable iff $\tilde{X}$ is metrizable.*
  (2) *(X, φ) is topologically transitive iff $(\tilde{X}, \tilde{\varphi})$ is topologically transitive.*
  (3) *(X, φ) has a dense set of periodic points iff the same is true of $(\tilde{X}, \tilde{\varphi})$.*
  (4) *(X, φ) is a minimal dynamical system iff the minimal homeomorphism extension has the same property.*
  (5) *The recurrent points in X are dense iff the recurrent points in $\tilde{X}$ are dense.*

Proof. (1) is routine.

(2) Let $(\tilde{X}, \tilde{\varphi})$ be topologically transitive, and $\emptyset \neq \mathcal{O} \subset X$. Then $\tilde{\mathcal{O}} := p^{-1}(\mathcal{O})$ is nonempty in $\tilde{X}$, so by assumption $\tilde{X} = \cup_{n=0}^{\infty} \tilde{\varphi}^{-n}(\tilde{\mathcal{O}})$. Let $x \in X$ and $\tilde{x} \in p^{-1}(x)$. Then there exists n such that $\tilde{x} \in \tilde{\varphi}^{-n}(\tilde{y})$, $\tilde{y} \in \tilde{\mathcal{O}}$. So $\tilde{y} = \tilde{\varphi}^n(\tilde{x})$, and so $p(\tilde{y}) = p(\tilde{\varphi}^n(\tilde{x}) = \varphi^n \circ p(\tilde{x}) = \varphi^n(x)$. Thus $x \in \varphi^{-n}(p(\tilde{y}) \subset \varphi^{-n}(\mathcal{O})$.

For the other direction, by Corollary 1 we can assume, without loss of generality, that $(\tilde{X}, \tilde{\varphi})$ is the canonical minimal homeomorphism

extension of (X, φ). The basic open sets in $\tilde{X}$ have the form

$$\tilde{\mathcal{O}} = \tilde{X} \cap [\mathcal{O}_1 \times \cdots \times \mathcal{O}_N \times \Pi_{n=N+1}^{\infty} X_n]$$

for some $N \in \mathbb{N}, \mathcal{O}_1, \ldots \mathcal{O}_N$ open sets in X, and where $X_n = X$ for all $n > N$.

If $\tilde{\mathcal{O}}$ is nonempty, there is a point $x \in \mathcal{O}_N$ such that $\varphi^j(x) \in \mathcal{O}_{N-j}$, $j = 1, \ldots, N - 1$. Hence by continuity of φ there is a neighborhood $U \subset \mathcal{O}_N$ such that $\varphi^j(U) \subset \mathcal{O}_{N-j}$, $j = 1, \ldots, N - 1$.

Let $\tilde{x} \in \tilde{X}$ be arbitrary, $\tilde{x} = (x_1, \ldots, x_N, \ldots)$. By the topological transitivity of (X, φ) we can find $n \in \mathbb{N}$ such that $\varphi^n(x_N) \in U$. Thus,

$$\begin{aligned}
\tilde{\varphi}^n(\tilde{x}) &= (\varphi^n(x_1), \ldots, \varphi^n(x_N), \ldots) \\
&\in \varphi^{N-1}(U) \times \ldots \varphi(U) \times U \times \Pi_{n=N+1}^{\infty} X_n \\
&\in \mathcal{O}_1 \times \cdots \times \mathcal{O}_{N-1} \times \mathcal{O}_N \times \Pi_{n=N+1}^{\infty} X_n
\end{aligned}$$

so that $\tilde{\varphi}^n(\tilde{x}) \in \tilde{\mathcal{O}}$ which finishes the proof.

(3) If the periodic points are dense in $\tilde{X}$, let $x \in X$ and $\tilde{x} \in p^{-1}(x)$. Then there is a net $\{\tilde{y}_n\} \subset \tilde{X}$ converging to x. By Lemma 3, $p(\tilde{y}_n)$ is periodic in X, and converges to x.

For the converse, note that if $x \in X$ is periodic, say of period n, then there is a point $\tilde{x} \in \tilde{X}$, $p(\tilde{x}) = x$ with $\tilde{x}$ periodic of period n. Indeed, if x has orbit $x, \varphi(x), \ldots, \varphi^{n-1}(x)$, then, setting $x_j = \varphi^{n+1-j}(x)$, $j = 1, \ldots, n$, take $\tilde{x}$ to be the point with coordinates

$$\tilde{x} = (x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n, \ldots).$$

If $\tilde{\mathcal{O}}$ is a basic open set in $\tilde{X}$, we use the argument in (2) to find an integer N and an open set U as in (2). Let $y \in U$ be periodic, and set $x = \varphi^{N-1}(y)$. The point $\tilde{x} \in \tilde{X}$ is periodic and belongs to $\tilde{\mathcal{O}}$.

(4) Assume $(\tilde{X}, \tilde{\varphi})$ is a minimal dynamical system, and $Y \subset X$ a nonempty closed, φ-invariant subset. Then $p^{-1}(Y)$ is a nonempty closed invariant subset of $\tilde{X}$, so $p^{-1}(Y) = \tilde{X}$. Thus, $Y = X$, and so (X, φ) is minimal.

Conversely, assume (X, φ) be a minimal dynamical system, and let $\mathcal{Y} = \{Y_i\}_{i \in I}$ be a maximal chain of closed invariant subsets of $\tilde{X}$, ordered by inclusion. Then

$$Y = \cap_{i \in I} Y_i$$

is the minimal element of the chain, hence Y has no proper invariant subset. As $Y \neq \emptyset$, $p(Y)$ is a nonempty invariant subset of X, so $p(Y) = X$. Taking $\psi = \varphi|_Y$, and $q = p|_Y$, we have that (Y, ψ) is a homeomorphism extension of (X, φ). While we do not know *a priori* that (Y, ψ) is a minimal homeomorphism extension of (X, φ), if (Y, ψ)

is not a minimal homeomorphism extension, there is an intermediate extension (Z, σ), as in the proof of Lemma 1. As ψ is a minimal homeomorphism on Y, σ is a minimal homeomorphism on Z. It follows that the minimal homeomorphism extension of (X, φ) which lies between X and Y is necessarily a minimal homeomorphism.

Since $(\tilde{X}, \tilde{\varphi})$ was the canonical minimal extension, and since any two minimal extensions are conjugate, it follows that the dynamical system $(\tilde{X}, \tilde{\varphi})$ is minimal.

(5) If the recurrent points in $\tilde{X}$ are dense, let U be any nonempty open set in X. Then there exists $\tilde{y} \in p^{-1}(U)$ which is recurrent. But then $y := p(\tilde{y}) \in U$ is recurrent.

Now assume (X, φ) has a dense set of recurrent. First we show that if $x \in X$ is recurrent, there is $\tilde{x} \in p^{-1}(x)$ which is recurrent. So, let $x \in X$ be recurrent, and let $\tilde{x} = (x_1, x_2, \dots) \in \tilde{X}$ be such that $p(\tilde{x}) = x$ (so $x = x_1$). By the compactness of X and a standard diagonalization argument, there is a subsequence $\{n_j\}$ of $\mathbb{N}$ and $y_i \in X$, $i = 1, 2, \dots$, such that

$$\lim_j \varphi^{n_j}(x_i) = y_i, \ i = 1, 2, \dots$$

and $y_1 = x_1$. Since $\varphi(x_{i+1}) = x_i$, the same relation holds for the y_i, and hence $\tilde{y} := (x_1, y_2, y_3, \dots) \in \tilde{X}$. Since

$$\lim_j \varphi^{n_j}(y_i) = \lim_j \varphi^{n_j - i + 1}(x_1) = \lim_j \varphi^{n_j}(x_i) = y_i,$$

$i = 1, 2, \dots$, this shows that $\tilde{y} \in p^{-1}(x)$ is recurrent.

Now, let $\tilde{\mathcal{O}} \subset \tilde{X}$ be a basic open set, and let U be an open set in X and $N \in \mathbb{N}$ be as in the proof of (2). Let $x_N \in U$ be recurrent; by the above assertion we can find $\tilde{x} = (x_1, \dots, x_N, \dots)$ which is recurrent in X, and by construction $\tilde{x}$ lies in $\tilde{\mathcal{O}}$.

$\square$

3. Representations of Semicrossed Products

For the moment we will take an abstract approach: Let (X, φ) be a dynamical system, and consider the algebra generated by $C(X)$ and a symbol U, where U satisfies the relation

$$(\ddagger) \quad fU = U(f \circ \varphi), \ f \in C(X).$$

The elements F of this algebra can be viewed as noncommutative polynomials in U,

$$F = \sum_{n=0}^{N} U^n f_n, \ f_n \in C(X), N \in \mathbb{N}.$$

Let us call this algebra $\mathcal{A}_0$.

In [11] we formed the Banach Algebra $\ell_1(\mathcal{A}_0)$ by providing a norm to elements F as above as $||F||_1 = \sum_{n=0}^{N} ||f_n||$ and then completing $\mathcal{A}_0$ in this norm. Either approach yields the same semicrossed product.

By a representation of $\mathcal{A}_0$ we will mean a homomorphism of $\mathcal{A}_0$ into the bounded operators on a Hilbert space, which is a $*$-representation when restricted to $C(X)$, viewed as a subalgebra of $\mathcal{A}_0$, and such that $\pi(U)$ is an isometry.

Fix a point $x \in X$ and, for convenience, set $x_1 = x$, $x_2 = \varphi(x)$, $x_3 = \varphi^2(x)$, ... Define a representation π_x of $\mathcal{A}_0$ on $\ell^2(\mathbb{N})$ by

$$\pi_x(f)(z_1, z_2, \dots) = (f(x_1)z_1, f(x_2)z_2, \dots),$$

with $(z_n)_{n=1}^{\infty} \in \ell^2(\mathbb{N})$ and

$$\pi_x(U)(z_1, z_2, \dots) = (0, z_1, z_2, \dots).$$

Observe this is a representation of $\mathcal{A}_0$ since

$$\pi_x(fU)(z_1, z_2, \dots) = (0, f(x_2)z_1, f(x_3)z_2, \dots)$$

and

$$\begin{aligned}
\pi_x(Uf \circ \varphi)(z_1, z_2, \dots) &= \pi_x(U)(f \circ \varphi(x_1)z_1, f \circ \varphi(x_2)z_2, \dots) \\
&= (0, f \circ \varphi(x_1)z_1, f \circ \varphi(x_2)z_2, \dots) \\
&= (0, f(x_2)z_1, f(x_3)z_2, \dots)
\end{aligned}$$

Let $(\tilde{X}, \tilde{\varphi})$ be the canonical homeomorphism extension. (cf definition 2 and Corollary 1.) We will consider $\mathcal{A}_0$ as embedded in $\tilde{\mathcal{A}}_0$, where $\tilde{\mathcal{A}}_0$ is the algebra generated by $C(\tilde{X})$ and $\tilde{U}$, satisfying the same relation ($\ddagger$). Let $\tilde{x} \in \tilde{X}$ and set $x = p(\tilde{x})$ where $p : \tilde{X} \to X$ is the map in diagram ($\dagger$).

For $f \in C(X)$, let $\tilde{f} \in C(\tilde{X})$, $\tilde{f} = f \circ p$, and for $F = \sum_{n=0}^{N} U^n f_n$, $f_n \in C(X)$, let $\tilde{F} = \sum_{n=0}^{N} \tilde{U}^n \tilde{f}_n$. Observe that

$$\pi_{\tilde{x}}(\tilde{F}) = \pi_x(F).$$

3.1. **Nest Representations.** For nonselfadjoint operator algebras, the representations which can play the role of the primitive representations in the case of C*-algebras are the *nest representations*. Recall, a representation π of an algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a nest representations if the lattice of subspaces invariant under π is linearly ordered.

Let (X, φ) be a dynamical system with φ a homeomorphism, and x a point in X which is aperiodic. This means $\tilde{\varphi}^n(\tilde{x}) \neq \tilde{x}$ for all $n \geq 1$.

Lemma 4. *The weak closure of $\pi(C(X))$ is a masa in $\mathcal{B}(\mathcal{H})$.*

Proof. It is enough to show that the operator e_n belongs to the weak closure, where e_n is the multiplication operator which is 1 in the nth coordinate and zero elsewhere. We can find $f_m \in C(X)$ satisfying

$$f_m(x_j) = \begin{cases} 1 & \text{for } j = n \\ 0 & \text{for } j \neq n, \ j \leq m \end{cases}$$

and f_m is real-valued, $0 \leq f_m \leq 1$. Indeed, this follows from the Tietze Extension Theorem.

As $\pi_x(f_m) \to e_n$ weakly, we have e_n in the weak closure of $\pi_x(C(X))$, and we are done. $\square$

Proposition 1. *Let (X, φ) be a dynamical system. If $x \in X$ is aperiodic, then π_x is a nest representation.*

Proof. Since the weak closure of $\pi(C(X))$ is a masa (Lemma 4), the closed subspaces $\mathcal{S}$ of $\ell^2(\mathbb{N})$ invariant under $\pi(C(X))$ are the vectors $\tilde{z} \in \ell^2(\mathbb{N})$ which are supported on a given subset of $\mathbb{N}$. If such a subspace is also invariant under $\pi(U)$ then it has the form

$$\mathcal{S} = \{\tilde{z} \in \ell^2(\mathbb{N}) : z_n = 0 \text{ for } n \leq N\}$$

for some $N \in \mathbb{N}$. But then the subspaces $\mathcal{S}$ are nested. $\square$

To periodic points we can associate another class of nest representations. Let $\tilde{x} \in \tilde{X}$ be periodic of period N, so $\tilde{x} = (x_1, x_2, \dots)$ with $x_{i+N} = x_i$ for $i \in \mathbb{N}$. Let $\pi : \mathcal{A} \to \mathcal{B}(\ell^2(N))$ by $\pi(f)(z_1, \dots, z_N) = (f(x_1)z_1, \dots, f(x_N)z_N)$ and $\pi(U)(z_1, \dots, z_N) = (z_N, z_1, \dots, z_{N-1})$.

For C* crossed products $\mathcal{B} := C(X) \rtimes_\psi \mathbb{Z}$ where ψ is a homeomorphism, we have the representations Π_x and $\Pi_{y,\lambda}$ for x aperiodic, y periodic, and $\lambda \in \mathbb{T}$ given as follows: Π_x acts on $\ell^2(\mathbb{Z})$, where $\Pi_x(U)$ is the bilateral shift (to the right), and

$$\Pi_x(f)(\xi_n) = (f(\psi^n(x))\xi_n) \quad n \in \mathbb{Z}, \ f \in C(X).$$

$\Pi_{y,\lambda}$ acts on the finite dimensional space $\ell^2(p)$, where p is the period of the orbit of y. $\Pi_{y,\lambda}(U)$ is a cyclic permutation along the (finite) orbit of y composed with multiplication by λ, and $\Pi_{y,\lambda}(f)$ acts like $\Pi_x(f)$ along the orbit of y. These representations correspond to the pure state extensions of the states on $C(X)$, $f \to f(x)$ in the cases where x is aperiodic or periodic, respectively, and so are irreducible. However, not all irreducible representations of $\mathcal{B}$ need be of this form. Nevertheless, Tomiyama has shown:

Proposition 2. *Every ideal of $\mathcal{B}$ is the intersection of those ideals of the form $ker(\Pi_x)$ and $ker(\Pi_{y,\lambda})$ (x aperiodic, y periodic, λ in $\mathbb{T}$) which contain it.*

This is Proposition 4.1 of [13].

Corollary 3. *If (Y, σ) is a dynamical system with σ a homeomorphism, then for $F \in C(Y) \rtimes_\sigma \mathbb{Z}$,*

$$||F|| = \max\{A, B\}$$

where

$$A = \sup\{||\Pi_x(F)|| : x \ \text{aperiodic}\}$$

and

$$B = \sup\{||\Pi_{y,\lambda}(F)|| : y \ \text{periodic}, \ \lambda \in \mathbb{T}.$$

Proof. Denote by $|| \cdot ||$ the crossed product norm, and by $|| \cdot ||_*$ the norm defined in the statement of the corollary. Let I be the ideal in $C(Y) \rtimes_\sigma \mathbb{Z}$ of all F with $||F||_* = 0$. Every ideal of the form $ker(\Pi_x)$ (x aperiodic) and $ker(\Pi_{y,\lambda})$ (y periodic, $\lambda \in \mathbb{T}$) contains I. Since the zero ideal also this property, it follows from Proposition 2 that $I = (0)$. $\square$

Lemma 5. *Let (X, φ) be a dynamical system with φ a homeomorphism, and let $y \in X$ be periodic. For $F \in C(X) \rtimes_\varphi \mathbb{Z}^+$, we have $||\pi_y(F)|| \geq \sup_{\lambda \in \mathbb{T}} ||\Pi_{y,\lambda}(F)||$.*

Proof. Since any $F \in C(X) \rtimes_\varphi \mathbb{Z}^+$ can be approximated by elements with finitely many nonzero Fourier coefficients, we can assume F has this property. Let y have period p, and we can assume $F = \sum_{n=0}^{kp} U^n f_n$, $f_n \in C(X)$, for some $k \in \mathbb{Z}^+$.

Let $\xi = (\xi_1, \dots, \xi_p) \in \mathbb{C}^p$ be any vector of norm 1, and fix $\lambda \in \mathbb{T}$. For $N \in \mathbb{N}$ define a vector $\eta \in \ell^2(\mathbb{N})$ of norm 1 by

$$\eta = (\eta_1, \dots, \eta_{Np}, 0, 0, \dots) \ \text{where} \ \eta_{i+jp} = \lambda^{N-j}\xi_i/\sqrt{N}, \ i = 1, \dots p, \ j = 0, \dots N-1.$$

Now, for $k \leq j < N$,

$$< \pi_y(F)\eta, e_{i+jp} >= \lambda^{j-k}/\sqrt{N} < \Pi_{y,\lambda}(F)\xi, e_i^p >$$

where e_n resp. e_n^p are standard basis vectors in $\ell^2(\mathbb{N})$, resp., in $\mathbb{C}^p$. Thus, if N/k is large, it follows that $||\pi_y(F)\eta||$ is close to $||\Pi_{y,\lambda}(F)\xi||$. This proves the lemma. $\square$

Lemma 6. *Let (X, φ) be a dynamical system with φ a homeomorphism, and $F \in C(X) \rtimes_\varphi \mathbb{Z}^+$. For any $x \in X$,*

$$||\Pi_x(F)|| = \sup\{||\pi_y(F)|| : \ y \in Orbit(x)\}.$$

Proof. Given $\epsilon > 0$, there is a vector $\xi \in \ell^2\mathbb{Z}$, $\xi = (\xi_n)_{n \in \mathbb{Z}}$ with only finitely many $\xi_n \neq 0$, and such that

$$||\Pi_x(F)|| \leq ||\Pi_x(F)\xi|| + \epsilon.$$

Suppose $\xi_n = 0$ for $n < -N$, for some $N \in \mathbb{Z}^+$. Let $y = \varphi^{-N}(x)$, and define a vector $\eta \in \ell^2(\mathbb{N})$ by: $\eta_j = \xi_{j-N-1}$, $j = 1, 2, \ldots$. Then $||\eta|| = 1$, and

$$||\pi_y(F)\eta|| = ||\Pi_x(F)\xi||.$$

The lemma now follows. $\qquad\square$

Corollary 4. *Let (X, φ) be a dynamical system, and $(\tilde{X}, \tilde{\varphi})$ a minimal homeomorphism extension, with $p : \tilde{X} \to X$ the continuous surjection for which the diagram (†) commutes. Let $F \in C(X) \rtimes_\varphi \mathbb{Z}^+$, and let $\tilde{x} \in \tilde{X}$. Then*

$$||\Pi_{\tilde{x}}(\tilde{F})|| = \sup\{||\pi_y(F)|| : \ y = p(\tilde{y}), \ for \ \tilde{y} \in Orbit(\tilde{x})\}.$$

Proof. Observe that for any $\tilde{y} \in \tilde{X}$, and $\xi \in \ell^2(\mathbb{N})$,

$$\pi_{\tilde{y}}(\tilde{F})\xi = \pi_y(F)\xi.$$

Now apply Lemma 6. $\qquad\square$

Definition 6. For a dynamical system (X, φ) (φ not necessarily a homeomorphism), a periodic point $y \in X$ and $\lambda \in \mathbb{T}$, we define $\pi_{y,\lambda}$ exactly like $\Pi_{y,\lambda}$ in the case where φ is a homeomorphism.

Remark 3. Since $\Pi_{y,\lambda}$ is irreducible, the same is true for $\pi_{y,\lambda}$, and in particular $\pi_{y,\lambda}$ is a nest representation.

Corollary 5. *Let (X, φ) be a dynamical system, $F \in C(X) \rtimes_\varphi \mathbb{Z}^+$. Then*

$$||F|| = \max\{A, B\}$$

where

$$A = \sup\{||\pi_x(F)|| : x \ aperiodic\}$$

and

$$B = \sup\{||\pi_{y,\lambda}(F)|| : y \ periodic, \ \lambda \in \mathbb{T}.$$

Proof. Note the constant "A" is the same as in Corollary 3 , and by Lemma 6 the constant "B" is the same as in Corolary 3. For $y \in X$ periodic and $\lambda \in \mathbb{T}$, we have by Lemma 5

$$\sup_{\lambda \in \mathbb{T}} ||\pi_{y,\lambda}(F)|| = \sup_{\lambda \in \mathbb{T}} ||Pi_{\tilde{y},\lambda}(\tilde{F})|| \leq ||\pi_y(F)|| \leq ||\Pi_y(F)||$$

where $\tilde{y} \in \tilde{X}$ is periodic and $p(\tilde{y}) = y$.

By Cor. II.8 of [11], $||F|| = \sup_{x \in X} ||\pi_x(F)||$. Thus, denoting the $\sup\{A, B\}$ by $||F||_*$, by Corollary 3 it follows that $||F||_*$ is the norm of F in the crossed product $C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z}$. On the other hand,

$$||F||_* \leq \sup_{x \in X} ||\pi_x(F)|| = ||F|| \leq \sup_{\tilde{x} \in \tilde{X}} ||\Pi_{\tilde{x}}(\tilde{F})||$$

and the last term is dominated by the norm of $\tilde{F}$ in the crossed product, since the norm there is given by the supremum over all covariant representations. $\qquad\square$

Theorem 2. *Let (X, φ) be a dynamical system, $F \in C(X) \rtimes_\varphi \mathbb{Z}^+$. Then*

$$||F|| = \sup\{||\pi(F)|| : \pi \text{ is an isometric covariant nest representation}\}.$$

Proof. Indeed, we have found a subclass of the isometric covariant nest representations, namely the $\pi_{y,\lambda}$ (y periodic, $\lambda \in \mathbb{T}$) and π_x (x aperiodic) which yield $||F||$. $\qquad\square$

From the above, we obtain the following

Theorem 3. *Let (X, φ) be a dynamical system, and $(\tilde{X}, \tilde{\varphi})$ its minimal homeomorphism extension. Then the embedding of the semicrossed product $C(X) \rtimes_\varphi \mathbb{Z}^+ \hookrightarrow C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z}$ into the crossed product is a completely isometric isomorphism.*

Corollary 6. *With notation as above, the semicrossed product $C(X) \rtimes_\varphi \mathbb{Z}^+$ is semisimple iff $C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z}^+$ is semisimple.*

Proof. This follows from part (5) of Theorem 1 and the main result of [3]. $\qquad\square$

If the crossed product is a simple C*-algebra, the crossed product is necessarily the C*-envelope. However, as we will now show, it is always the case that the crossed product is the C*-envelope, even if it is not simple.

Lemma 7. *Let (X, φ) be a dynamical system, and $C(X) \rtimes_\varphi \mathbb{Z}^+$ the associated semicrossed product. Then the endomorphism α of $C(X)$, $\alpha(f) = f \circ \varphi$, extends to an endomorphism, again denoted by α, of the semicrossed product.*

Proof. Embed $C(X) \rtimes_\varphi \mathbb{Z}^+ \hookrightarrow C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z}$. The element U, which is an isometry in the semicrossed product, embeds to a unitary in the crossed product, and one can define an automorphism $\tilde{\alpha}$ on the crossed product, which extends the automorphism, also denoted by $\tilde{\alpha}$ of $C(\tilde{X})$, $\tilde{\alpha}(f) = f \circ \tilde{\varphi}$.

This is as follows: in $C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z}$ one has $U^* f U = f \circ \tilde{\varphi}$. For F an element of the crossed product, define $\tilde{\alpha}(F) = U^* F U$. Note that, if $\{f_n\}$ are the Fourier coefficients of F, then $\{f_n \circ \tilde{\varphi}\}$ are the Fourier coefficients of $\tilde{\alpha}(F)$.

In particular, if F belongs to the semicrossed product, and so its Fourier coefficients belong to the subalgebra $C(X) \hookrightarrow C(\tilde{X})$, then the

Fourier coefficients of $\tilde{\alpha}(F)$ also belong to the subalgebra $C(X)$, since for $f \in C(X)$, $\tilde{\alpha}(f) = \alpha(f)$. Thus, if we denote this map of $C(X) \rtimes_\varphi \mathbb{Z}^+$ by α, it is an endomorphism of the semicrossed product extending the endomorphism α of $C(X)$. □

Lemma 8. *Let (X, φ) be a dynamical system, and embed $\mathcal{A} := C(X) \rtimes_\varphi \mathbb{Z}^+ \hookrightarrow C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z}$. Then*

$$\cup_{k=0}^\infty \tilde{\alpha}^{-k}(\mathcal{A}) \subset C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z} \text{ is a dense subalgebra.}$$

Proof. It follows from the proof of Corollary 2 that, viewing $C(X)$ as embedded in $C(\tilde{X})$, that $\cup_{n=0}^\infty \tilde{\alpha}^{-n}(C(X))$ is a dense subalgebra of $C(\tilde{X})$.

Given $F \in C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z}$ and $\epsilon > 0$, there is $G \in C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z}$ with finitly many nonzero Fourier coefficients, say $G = \sum_{n=0}^N U^n g_n$, with $||F - G|| < \epsilon$. By the first paragraph, for each g_n there is an h_n in the dense subalgebra of $C(\tilde{X})$ with $||g_n - h_n|| < \frac{\epsilon}{N+1}$. But if $H = \sum_{n=0}^N U^n h_n$, we have $||F - H|| < 2\epsilon$, and $H \in \cup_{k=0}^\infty \tilde{\alpha}^{-k}(\mathcal{A})$. □

Theorem 4. *Let (X, φ) be a dynamical system, and $(\tilde{X}, \tilde{\varphi})$ its minimal homeomorphism extension. Then the C*-envelope of the semicrossed product $C(X) \rtimes_\varphi \mathbb{Z}^+$ is the crossed product $C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z}$.*

Proof. By Theorem 3 the embedding

$$C(X) \rtimes_\varphi \mathbb{Z}^+ \hookrightarrow C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z}$$

is completely isometric. Suppose there is a C*-algebra $\mathcal{B}$, a completely isometric embedding $\iota : C(X) \rtimes_\varphi \mathbb{Z}^+ \to \mathcal{B}$, and a surjective C*-homomorphism $q : C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z} \to \mathcal{B}$.

If q is not an isomorphism, let $0 \neq F \in \ker(q)$. Assume $||F|| = 1$. By Lemma 8 there is an element $G = \sum_{n=0}^N U^n g_n$ with $g_n \in \cup_{n=0}^\infty \tilde{\alpha}^{-n}(C(X))$, viewing $C(X)$ as a subalgebra of $C(\tilde{X})$, and such that $||F - G|| < \frac{1}{2}$. In particular, there is $m \in \mathbb{Z}^+$ such that $g_n \circ \tilde{\varphi}^m \in C(X)$, $0 \le n \le N$.

Now $GU^m = \sum_{n=0}^N g_n \circ \tilde{\varphi}^m \in C(X) \rtimes_\varphi \mathbb{Z}^+$ and $||GU^m|| = ||G|| > \frac{1}{2}$. On the other hand, since $q(FU^m) = q(F)q(U^m) = 0$ we have

$$||q(GU^m)|| = ||q(GU^m - FU^m)|| \le ||GU^m - FU^m|| \le ||G - F|| < \frac{1}{2}.$$

This contradiction shows that $\ker(q) = (0)$, and hence that $C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z}$ is the C* envelope of the semicrossed product. □

Finally, we make use of the relation between properties of dynamical systems and their extensions to obtain

Proposition 3. *Let (X, φ) be a dynamical system. If the C^*-envelope of the semicrossed product is a simple C^*-algebra, then $C(X) \rtimes_\varphi \mathbb{Z}^+$ is semi-simple.*

Remark 4. The converse is false.

Proof. By Theorem 4, the C^*-envelope is a crossed product, $C(\tilde{X}) \rtimes_{\tilde{\varphi}} \mathbb{Z}$, where $(\tilde{X}, \tilde{\varphi})$ is the (unique) minimal homeomorphism extension of (X, φ). As is well known (e.g. [10] Proposition 7.9.6), the crossed product is simple if and only if the dynamical system $(\tilde{X}, \tilde{\varphi})$ is minimal; i.e., every point has a dense orbit. By Theorem 1, this is equivalent to the condition that (X, φ) is minimal. In particular, the system (X, φ) is recurrent; so by [3] it follows that the semicrossed product is semi-simple. $\qquad\qquad\square$

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